Qualifying Exam   Complex Analysis   August, 2022

Note: All statements require proofs. You can make references to standard theorems; however, you must either state the name of the theorem, if it is a well-known theorem, or state the relevant part of the theorem. For example, "by the monotone convergence theorem," or, "we learned that the integrals of an increasing sequence of positive functions converge to the integral of their limit," are good references but, "by a convergence theorem the integrals converge," is **not** a good reference.

1. Prove that, for any $r \in (0,1)$, there is $N \in \mathbb{N}$ such that the polynomial equation
$$\sum_{n=0}^{N} \frac{z^n}{n!} = e$$
has a unique solution in the disk $D(1,r) = \{z \in \mathbb{C} : |z-1| < r\}$.

2. Prove that the infinite series
$$\sum_{n=1}^{\infty} \frac{e^{nz}}{(z-n)^2}$$
converges at every $z \in H := \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$, and the limit, denoted by $f(z)$, is an analytic function on H.

3. Compute
$$\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - \theta) d\theta.$$
Hint: Express $e^{e^{i\theta}}$ in the rectangular form.

4. Prove the Fundamental Theorem of Algebra: Every non-constant polynomial with complex coefficients has a complex root.

5. Let f be analytic in $\{z \in \mathbb{C} : 0 < |z - z_0| < 1\}$, where $z_0 \in \mathbb{C}$. Suppose z_0 is a pole of f. Prove that for any $r \in (0,1)$, there is $R > 0$ such that
$$\{w \in \mathbb{C} : |w| > R\} \subset f(\{z \in \mathbb{C} : 0 < |z| < r\}).$$